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Solution of fully developed free convection of a micropolar fluid in a vertical channel by homotopy analysis method

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SUMMARY

In this paper, we reconsider the problem of fully developed natural convection heat and mass transfer of a micropolar fluid in a vertical channel with asymmetric wall temperatures and concentrations. The resulting boundary-value problem is solved analytically by the homotopy analysis method. The accuracy of the present solution is found to be in excellent agreement with the solutions of Cheng (*Int. Commun. Heat Mass Transfer* 2006; **33**:627–635). Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: heat and mass transfer; micropolar fluid; homotopy analysis method

1. INTRODUCTION

The phenomenon of convective heat and mass transfer has been intensively investigated due to many industrial applications, such as air conditioning of a room, material processing, cooling of nuclear reactors and many other practical situations. The natural convection heat transfer of liquid metals (with additives/suspensions) in a confined enclosure is considered as an important problem to passive cooling system. Fluids with additives/suspensions are considered as non-Newtonian, which can be fully described by the theory of micropolar fluids first developed by Erigen [1]. Chamkha *et al.* [2] were the first to study the fully developed free convection of a micropolar fluid in a vertical channel. This study was further extended by Cheng [3] to include mass transfer. The solutions to the boundary-value problems in [2, 3] can be given in closed-forms. However, it would be difficult to find closed-form solution for complex problems.

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The purpose of this paper is to present an accurate numerical solution of the boundary-value problem given in [3] using the analytic homotopy analysis method (HAM) first developed by Liao [4]. This method has been successfully applied to many non-linear problems, cf. [5–18]. Numerical results are presented graphically and are found to be in excellent agreement with the solutions of Cheng [3].

2. BASIC EQUATIONS

Consider the free convection of a micropolar fluid between two vertical plates, the space between the plates being b. The flow is assumed laminar, steady and fully developed, i.e. the transverse velocity is zero. It is also assumed that the walls are heated uniformly, but their temperatures may be different resulting in an asymmetric heating situation. The dimensionless governing equations are [3]:

$$(1+R)\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}+R\frac{\mathrm{d}H}{\mathrm{d}Y}=-\theta-N\phi \tag{1}$$

$$\left(1+\frac{R}{2}\right)\frac{\mathrm{d}^2H}{\mathrm{d}Y^2}-BR\left(2H+\frac{\mathrm{d}U}{\mathrm{d}Y}\right)=0 \tag{2}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}Y^2}=0 \tag{3}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}Y^2}=0 \tag{4}$$

subject to the boundary conditions

$$U(0)=0, \quad H(0)=0, \quad \theta(0)=1, \quad \phi(0)=1 \tag{5}$$

$$U(1)=0, \quad H(1)=0, \quad \theta(1)=m, \quad \phi(1)=n \tag{6}$$

where U is the dimensionless vertical velocity component, θ the dimensionless temperature, H the dimensionless microrotation, ϕ the dimensionless concentration, Y the dimensionless transverse coordinate, m the wall temperature ratio, n the wall concentration ratio and B and R are the material parameters.

Solving Equations (3) and (4) subject to boundary conditions (5) and (6) give the dimensionless temperature and concentration as

$$\theta=(m-1)Y+1 \tag{7}$$

$$\phi=(n-1)Y+1 \tag{8}$$

Now substituting (7) and (8) into (1) and rewriting (2), then we have

$$\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}+\frac{R}{1+R}\frac{\mathrm{d}H}{\mathrm{d}Y}=-\frac{1}{1+R}[(m-1)Y+1)-N((n-1)Y+1] \tag{9}$$

$$\frac{\mathrm{d}^2H}{\mathrm{d}Y^2}-\frac{2BR}{2+R}\left(2H+\frac{\mathrm{d}U}{\mathrm{d}Y}\right)=0 \tag{10}$$

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subject to the boundary conditions

$$U(0)=0, \quad H(0)=0 \tag{11}$$

$$U(1)=0, \quad H(1)=0 \tag{12}$$

3. SOLUTION BY HAM

In this section, we apply the HAM to solve system (9)–(12). We assume that the solutions of $U(Y)$ and $H(Y)$ can be expressed by a set of base functions

$$\{Y^n | n=0,1,2,\ldots\}$$

in the following forms:

$$U(Y)=\sum_{n=0}^{+\infty} a_n Y^n, \quad H(Y)=\sum_{n=0}^{+\infty} b_n Y^n \tag{13}$$

where a_n and b_n are coefficients. This provides us with the *first rule of solution expression* of (9)–(12). Under the rule of solution expression and according to the boundary conditions (11) and (12), it is straightforward to choose

$$U_0(Y)=H_0(Y)=0$$

as the initial boundary approximations of $U(Y)$ and $H(Y)$, and to choose the auxiliary linear operator

$$\mathscr{L}[\phi_i(Y;q)]=\frac{\partial^2 \phi_i(Y;q)}{\partial Y^2}, \quad i=1,2$$

with the property

$$\mathscr{L}[C_{1,i}+YC_{2,i}]=0$$

where C_i $(i=1,2)$ are constants of integration. The system of non-linear operators in HAM for the present problem is as follows:

$$\mathscr{N}_1[\phi_i(Y;q)]=\frac{\partial^2 \phi_1(Y;q)}{\partial Y^2}+\frac{R}{1+R}\frac{\partial \phi_2(Y;q)}{\partial Y} + \frac{1}{1+R}[(m-1)Y+1)-N((n-1)Y+1]$$

$$\mathscr{N}_2[\phi_i(Y;q)]=\frac{\partial^2 \phi_2(Y;q)}{\partial Y^2}-\frac{2BR}{2+R}\left(2\phi_2(Y;q)+\frac{\partial \phi_1(Y;q)}{\partial Y}\right)$$

Using the above definition, we construct the *zeroth-order deformation equation*

$$(1-q)\mathscr{L}[\phi_1(Y;q)-U_0(Y)]=q\hbar\mathscr{N}_1[\phi_i(Y;q)] \tag{14}$$

$$(1-q)\mathscr{L}[\phi_2(Y;q)-H_0(Y)]=q\hbar\mathscr{N}_2[\phi_i(Y;q)] \tag{15}$$

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Obviously, when $q=0$ and 1,

$$\phi_1(Y;0)=U_0(Y), \quad \phi_1(t;1)=U(Y)$$

$$\phi_2(Y;0)=H_0(Y), \quad \phi_2(t;1)=H(Y)$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi_1(Y;q)$ and $\phi_2(Y;q)$ vary from the initial guesses, $U_0(Y)$ and $H_0(Y)$, to the solutions, $U(Y)$ and $H(Y)$. Expanding $\phi_1(Y,q)$ and $\phi_2(Y,q)$ in Taylor series with respect to q one has

$$\phi_1(Y;q)=U_0(Y)+\sum_{m=1}^{+\infty} U_m(Y)q^m \tag{16}$$

$$\phi_2(Y;q)=H_0(Y)+\sum_{m=1}^{+\infty} H_m(Y)q^m \tag{17}$$

where

$$U_m(Y)=\frac{1}{m!}\frac{\partial^m \phi_1(Y;q)}{\partial q^m}\bigg|_{q=0}$$

$$H_m(Y)=\frac{1}{m!}\frac{\partial^m \phi_2(Y;q)}{\partial q^m}\bigg|_{q=0}$$

Now define the vectors

$$\mathbf{U}_n=\{U_0(Y),U_1(Y),\ldots,U_n(Y)\}$$

$$\mathbf{H}_n=\{H_0(Y),H_1(Y),\ldots,H_n(Y)\}$$

Differentiating the zeroth-order deformation equations (14) and (15) m times with respect to q, and finally dividing by $m!$, we get the mth-order deformation equations as

$$\mathscr{L}[U_m(Y)-\chi_m U_{m-1}(Y)]=\hbar R_m(\mathbf{U}_m) \tag{18}$$

$$\mathscr{L}[H_m(Y)-\chi_m H_{m-1}(Y)]=\hbar R_m(\mathbf{H}_m) \tag{19}$$

subject to boundary conditions

$$U_m(Y)=H_m(Y)=0 \tag{20}$$

where

$$\mathscr{R}_{1,m}(\mathbf{U}_{m-1})=U''_{m-1}(Y)+\frac{R}{1+R}H'_{m-1}$$
$$+(1-\chi_m)\frac{1}{1+R}[(m-1)Y+1)-N((n-1)Y+1]$$

$$\mathscr{R}_{2,m}(\mathbf{H}_{m-1})=H''_{m-1}(Y)-\frac{2BR}{2+R}(2H_{m-1}+U'_{m-1})$$

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where the prime denotes differentiation with respect to Y. Now the solutions of the mth-order deformation Equations (18) and (19) for $m\geqslant 1$ become

$$U_m(Y)=\chi_m U_{m-1}(Y)+\hbar\int_0^x\int_0^x \mathscr{R}_{1,m}(\mathbf{U}_{m-1})\,\mathrm{d}\tau\,\mathrm{d}\tau+C_{1,1}+YC_{2,1} \tag{21}$$

$$H_m(Y)=\chi_m H_{m-1}(Y)+\hbar\int_0^x\int_0^x \mathscr{R}_{2,m}(\mathbf{H}_{m-1})\,\mathrm{d}\tau\,\mathrm{d}\tau+C_{1,2}+YC_{2,2} \tag{22}$$

where the constants of integration $C_{i,1}, C_{i,2}$ $(i=1,2)$ are determined by the boundary conditions (20), and χ_m is

$$\chi_m=\begin{cases}0, & n\leqslant 1\\ 1, & n>1\end{cases}$$

We now successively obtain

$$U_1(Y)=\frac{\hbar}{6(R+1)}(Y^2-Y)[-Y+m(Y+1)+N(nY+n-Y+2)+2]$$

$$U_2(Y)=\frac{\hbar(\hbar+1)}{6(R+1)}(Y^2-Y)[-Y+m(Y+1)+N(nY+n-Y+2)+2]$$

$$\vdots$$

$$H_1(Y)=0$$

$$H_2(Y)=-\frac{BR\hbar^2}{12(R^2+3R+2)}(Y^2-Y)[-Y^2+3t+m(Y^2+Y-1)$$
$$+N(-Y^2+3Y+n(Y^2+Y-1)-1)-1]$$

$$\vdots$$

Then the series solution to system (9)–(12) is

$$U(Y)=\sum_{n=0}^{+\infty}a_n\hbar Y^n,\quad H(Y)=\sum_{n=0}^{+\infty}b_n\hbar Y^n \tag{23}$$

The dimensionless volume flow rate, Q, the dimensionless total heat rate added to the fluid, E, and the dimensionless total species rate added to the fluid are given, respectively, as

$$Q=\int_0^1\sum_{n=0}^{+\infty}a_n\hbar Y^n\,\mathrm{d}Y \tag{24}$$

$$E=\int_0^1\left(\sum_{n=0}^{+\infty}a_n\hbar Y^n\right)\theta\,\mathrm{d}Y \tag{25}$$

$$\Phi=\int_0^1\left(\sum_{n=0}^{+\infty}a_n\hbar Y^n\right)\phi\,\mathrm{d}Y \tag{26}$$

4. RESULTS AND DISCUSSION

We note that the series solutions (23) contain the auxiliary parameter $\hbar$. The validity of the method is based on such an assumption that the series (16) and (17) converge at $q=1$. It is the auxiliary parameter $\hbar$ that ensures that this assumption can be satisfied. In general, by means of the so-called $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$, which ensures that the solution series is convergent. We can investigate the influence of $\hbar$ on the residual error defined as

$$\text{Residual Error for } U=U''(Y)+\frac{R}{1+R}H'(Y)+\frac{1}{1+R}[(m-1)Y+1)-N((n-1)Y+1]$$

$$\text{Residual Error for } H=H''(Y)-\frac{2BR}{2+R}[2H+U'(Y)]$$

Based on Figures 1 and 2, the valid regions of $\hbar$ correspond to the line segments nearly parallel to the horizontal axis. Figures 3 and 4 show the residual errors of the 10th-order HAM solutions

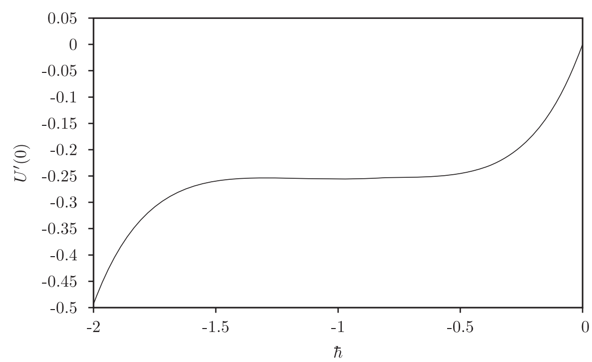

Figure 1. The $\hbar$-curves obtained from the fifth-order HAM approximation solution for U in the case $m=0.6$, $n=0.3$, $N=2$, $B=1$ and $R=1$.

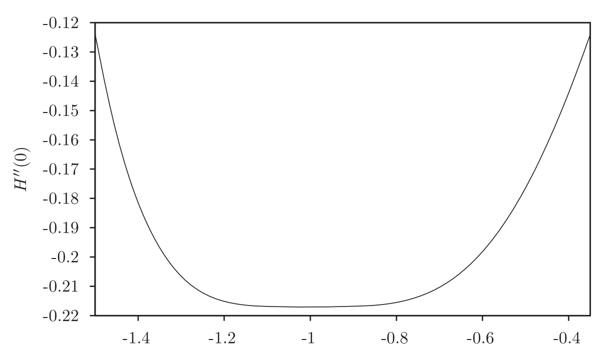

Figure 2. The $\hbar$-curves obtained from the fifth-order HAM approximation solution for H in the case $m=0.6$, $n=0.3$, $N=2$, $B=1$ and $R=1$.

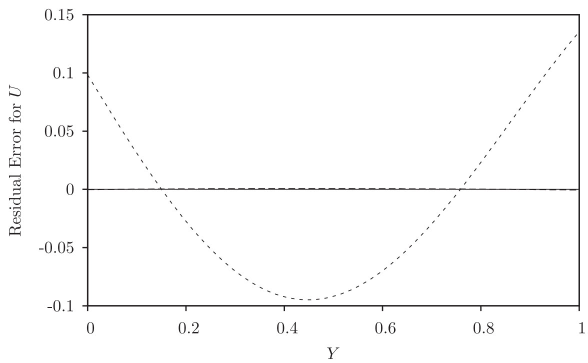

Figure 3. Error of 10th-order HAM approximation solution for U in the case $m=0.6$, $n=0.3$, $N=2$, $B=1$ and $R=1$, solid line: $\hbar=-1$; dotted line: $\hbar=-0.6$; dashed line: $\hbar=-1.4$.

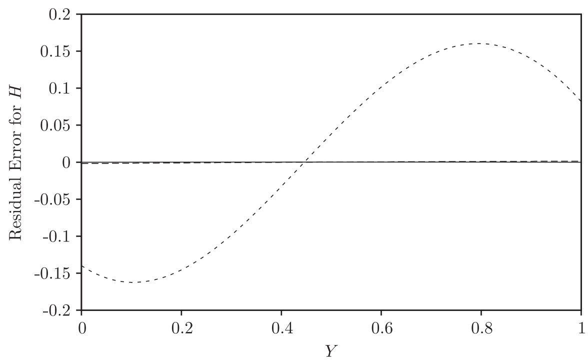

Figure 4. Error of 10th-order HAM approximation solution for H in the case $m=0.6$, $n=0.3$, $N=2$, $B=1$ and $R=1$, solid line: $\hbar=-1$; dotted line: $\hbar=-0.6$; dashed line: $\hbar=-1.4$.

for several values of $\hbar$. Taking $\hbar=-1$, the three-term approximate solutions for U and H are

$$
\begin{aligned}
U(Y) \sim\ & \frac{(m-1-N+nN)BR^2}{60(R+1)^2(R+2)}Y^5+\frac{(1+N)BR^2}{12(R+1)^2(R+2)}Y^4 \\
& -\frac{(2+m+2N+nN)BR^2}{18(R+1)^2(R+2)}Y^3+\frac{1-m-nN+N}{6(R+1)}Y^3 \\
& +\frac{(1+m+N+nN)BR^2}{24(R+1)^2(R+2)}Y^2-\frac{1+N}{2(R+1)}Y^2 \\
& +\frac{(1-m+N-nN)BR^2}{360(R+1)^2(R+2)}Y+\frac{2+m+nN+2N}{6(R+1)}Y
\end{aligned}
$$

$$
\begin{aligned}
H(Y) \sim\ & \frac{(1-m+N-nN)B^2R^2}{90(R+1)(R+2)^2}Y^6 - \frac{(1+N)B^2R^2}{15(R+1)(R+2)^2}Y^5 \\
& + \frac{(1-m+N-nN)BR}{12R^2+3R+2}Y^4 + \frac{(2+m+2N+nN)B^2R^2}{18(R+1)(R+2)^2}Y^4 \\
& - \frac{(1+N)BR}{3(R^2+3R+2)}Y^3 - \frac{(1+m+N+nN)B^2R^2}{18(R+1)(R+2)^2}Y^3 \\
& + \frac{(2+m+2N+nN)BR}{3(R^2+3R+2)}Y^2 - \frac{(1+m+N+nN)BR}{12(R^2+3R+2)}Y \\
& + \frac{(m+nN)B^2mR^2}{90(R+1)(R+2)^2}Y
\end{aligned}
$$

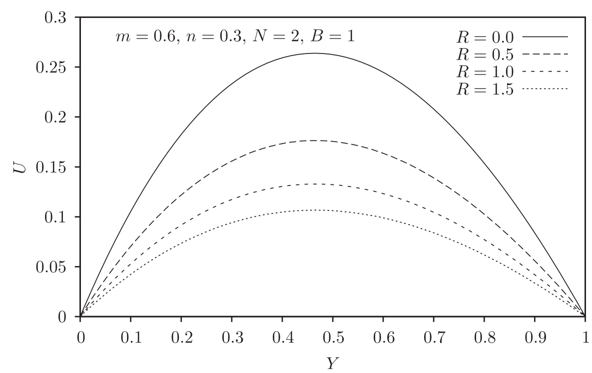

Figure 5. Effects of vortex viscosity parameter on the velocity profiles.

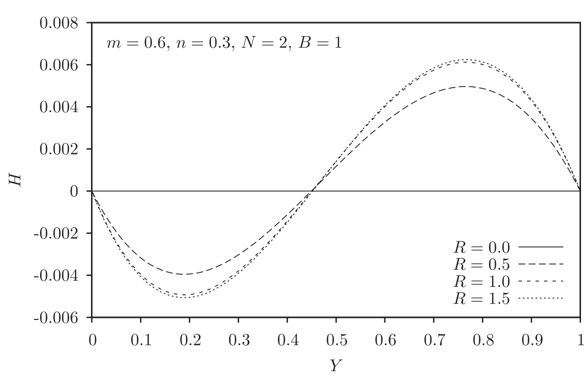

Figure 6. Effects of vortex viscosity parameter on the microrotation profiles.

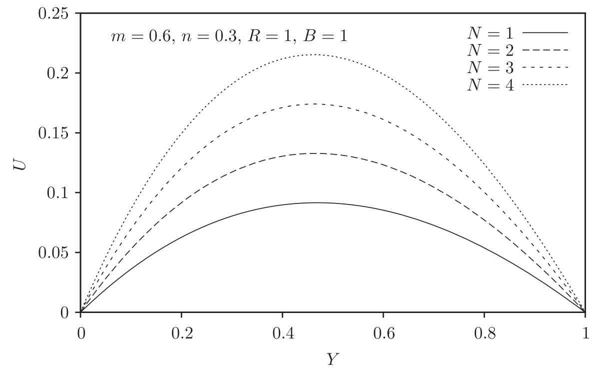

Figure 7. Effects of buoyancy ratio on the velocity profiles.

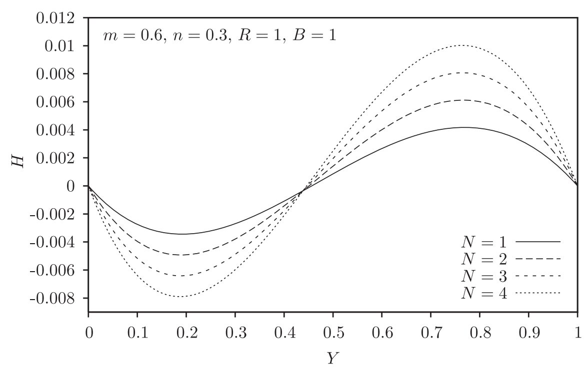

Figure 8. Effects of buoyancy ratio on the microrotation profiles.

We plot in Figures 5 and 6 the velocity and microrotation profiles given by the fifth-order HAM approximations for various vortex viscosity parameters $R=0, 0.5, 1, 1.5$, and the case $m=0.6$, $n=0.3$, $N=2$, $B=1$. The effect of the buoyancy ratio N on the velocity and microrotation profiles obtained from the fifth-order HAM approximations is shown in Figures 7 and 8 for the case $m=0.6, n=0.3$, $R=1$, $B=1$.

Figures 9–11 show the effects of the buoyancy ratio R and vortex viscosity parameter N on the volume flow rate Q, total heat rate added to the fluid E and total species rate added to the fluid Φ, respectively, for the case $m=0.6, n=0.3$ and $B=1$.

5. CONCLUSIONS

In this paper, we presented numerical solution of the boundary-value problem for free convection based on the homotopy analysis method (HAM). The HAM solutions are in excellent agreement

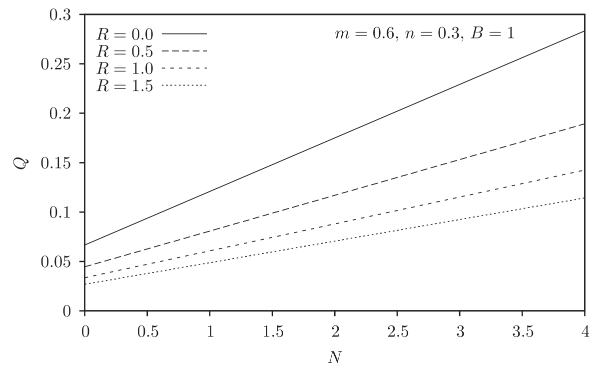

Figure 9. Effects of buoyancy ratio and vortex viscosity parameter on the volume flow rate.

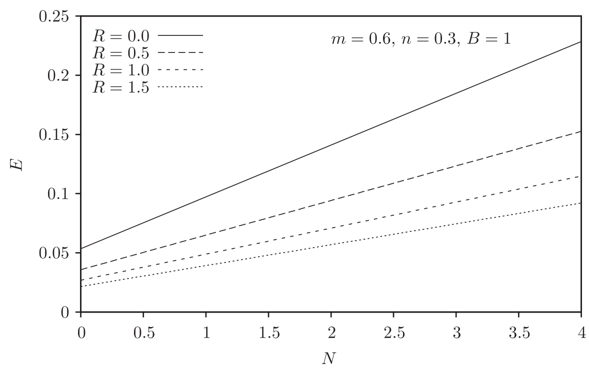

Figure 10. Effects of buoyancy ratio and vortex viscosity parameter on the total heat rate added to the fluid.

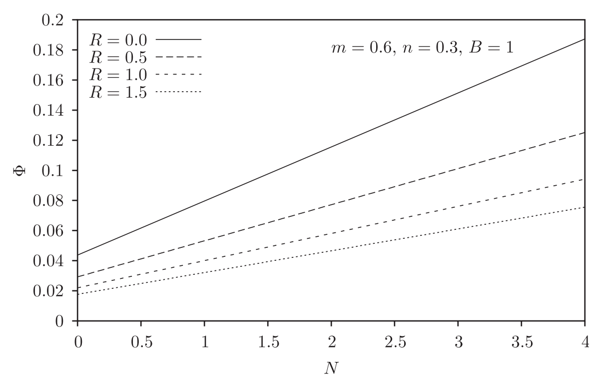

Figure 11. Effects of buoyancy ratio and vortex viscosity parameter on the total species rate added to the fluid.

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with the solutions of Cheng [3]. We believe that the HAM can be a useful tool in the analysis of more complicated problems.

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